4.1 Centre of Gravity

Everybody is attracted towards the centre of the earth due gravity. The force of attraction is proportional to mass of the body. Everybody consists of innumerable particles, however the entire weight of a body is assumed to act through a single point and such a single point is called centre of gravity.

Every body has one and only centre of gravity.

4.2 Centroid

In case of plane areas (bodies with negligible thickness) such as a triangle quadrilateral, circle etc., the total area is assumed to be concentrated at a single point and such a single point is called centroid of the plane area.

The term centre of gravity and centroid has the same meaning but the following differences.

1. Centre of gravity refer to bodies with mass and weight whereas, centroid refers to plane areas.
2. Centre of gravity is a point is a point in a body through which the weight acts vertically downwards irrespective of the position, whereas the centroid is a point in a plane area such that the moment of areas about an axis through the centroid is zero

![Diagram of Plane area 'A' with centroid G]

Note: In the discussion on centroid, the area of any plane figure is assumed as a force equivalent to the centroid referring to the above figure G is said to be the centroid of the plane area A as long as

\[ a_1d_1 - a_2d_2 = 0. \]
4.3 Location of centroid of plane areas

The position of centroid of a plane area should be specified or calculated with respect to some reference axis i.e. X and Y axis. The distance of centroid G from vertical reference axis or Y axis is denoted as X and the distance of centroid G from a horizontal reference axis or X axis is denoted as Y.

While locating the centroid of plane areas, a bottommost horizontal line or a horizontal line through the bottommost point can be made as the X – axis and a leftmost vertical line or a vertical line passing through the leftmost point can be made as Y- axis.
In some cases the given figure is symmetrical about a horizontal or vertical line such that the centroid of the plane area lies on the line of symmetry.

The above figure is symmetrical about a vertical line such that G lies on the line of symmetry. Thus
X = b/2.
Y = ?

The centroid of plane geometric area can be located by one of the following methods
   a) Graphical methods
   b) Geometric consideration
   c) Method of moments

The centroid of simple elementary areas can be located by geometric consideration. The centroid of a triangle is a point, where the three medians intersect. The centroid of a square is a point where the two diagonals bisect each other. The centroid of a circle is centre of the circle itself.

4.4 METHOD OF MOMENTS TO LOCATE THE CENTROID OF PLANE AREAS
Let us consider a plane area A lying in the XY plane. Let G be the centroid of the plane area. It is required to locate the position of centroid G with respect to the reference axis like Y-axis and Xi-axis i.e., to calculate X and Y. Let us divide the given area A into smaller elemental areas a₁, a₂, a₃ ....... as shown in figure. Let g₁, g₂, g₃...... be the centroids of elemental areas a₁, a₂, a₃ ....... etc.

Let x₁, x₂, x₃ etc be the distance of the centroids g₁ g₂ g₃ etc. from Y-axis is =A X --(1)
The sum of the moments of the elemental areas about Y-axis is
a₁ . x₁ + a₂ . x₂ + a₃ . x₃ + .................(2)

Equating (1) and (2)

\[
A . \bar{X} = a₁ . x₁ + a₂ . x₂ + a₃ . x₃ + .......... \\
\bar{X} = \frac{a₁ . x₁ + a₂ . x₂ + a₃ . x₃ + ..........}{A} \\
\bar{X} = \sum \frac{(ax)}{A} \text{ or } \bar{X} = \frac{\int x.dA}{A}
\]

Where a or dA represents an elemental area in the area A, x is the distance of elemental area from Y-axis.

Similarly

\[
\bar{Y} = \frac{a₁ . y₁ + a₂ . y₂ + a₃ . y₃ + ..........}{A} \\
\bar{Y} = \sum \frac{(a.y)}{A} \text{ or } \bar{Y} = \frac{\int y.dA}{A}
\]
TO LOCATE THE CENTROID OF A RECTANGLE FROM THE FIRST PRINCIPLE
(METHOD OF MOMENTS)

Let us consider a rectangle of breadth b and depth d. Let g be the centroid of the rectangle. Let us consider the X and Y axis as shown in the figure.

Let us consider an elemental area dA of breadth b and depth dy lying at a distance of y from the X axis.
W.K.T

\[ \bar{Y} = \frac{\int_{0}^{d} y \cdot dA}{A} \]

\[ A = b \cdot d \]
\[ dA = b \cdot dy \]

\[ \bar{Y} = \frac{\int_{0}^{d} y \cdot (b \cdot dy)}{b \cdot d} \]

Similarly

\[ \bar{X} = \frac{\int_{0}^{b} x \cdot dA}{A} \]
\[ A = b \cdot d \]
\[ dA = dx \cdot d \]

\[ \bar{X} = \frac{\int_{0}^{b} x \cdot (dx \cdot d)}{b \cdot d} \]

\[ \bar{Y} = \frac{1}{d} \int_{0}^{d} y \cdot dy \]

\[ \bar{Y} = \frac{1}{d} \left[ \frac{y^2}{2} \right]_{0}^{d} \]

\[ \bar{Y} = \frac{1}{d} \left[ \frac{d^2}{2} \right] \]

\[ \bar{Y} = \frac{d}{2} \]

\[ \bar{X} = \frac{1}{b} \left[ \frac{x^2}{2} \right]_{0}^{b} \]

\[ \bar{X} = \frac{1}{b} \left[ \frac{b^2}{2} \right] \]

\[ \bar{X} = \frac{b}{2} \]
Centroid of a triangle

Let us consider a right angled triangle with a base $b$ and height $h$ as shown in figure. Let $G$ be the centroid of the triangle. Let us consider the $X$-axis and $Y$-axis as shown in figure.

Let us consider an elemental area $dA$ of width $b_1$ and thickness $dy$, lying at a distance $y$ from $X$-axis.

W.K.T

\[
\bar{Y} = \frac{\int_{0}^{h} y \, dA}{A}
\]

\[
A = \frac{bh}{2}
\]

\[
dA = b_1 \cdot dy
\]

\[
\bar{Y} = \frac{\int_{0}^{h} y \cdot (b_1 \cdot dy)}{\frac{b \cdot h}{2}} \quad \text{[as $x$ varies $b_1$ also varies]}
\]
\[
\bar{Y} = \frac{2}{h} \int_0^h \left( y - \frac{y^2}{h} \right) dy \\
\bar{Y} = \frac{2}{h} \left[ \frac{y^2}{2} - \frac{y^3}{3h} \right]_0^h \\
\bar{Y} = \frac{2}{h} \left[ \frac{h^2}{2} - \frac{h^3}{3h} \right] \\
\bar{Y} = 2h \left| \frac{1}{2} - \frac{1}{3} \right| \\
\bar{Y} = \frac{2h}{6} \\
\bar{Y} = \frac{h}{3} \text{ similarly } \bar{X} = \frac{b}{3}
\]
Centroid of a semi-circle

Let us consider a semi-circle, with a radius $r$.
Let $O$ be the centre of the semi-circle. Let $G$ be centroid of the semi-circle. Let us consider the $x$ and $y$ axes as shown in figure.

Let us consider an elemental area $dA$ with centroid $g$ as shown in fig. Neglecting the curvature, the elemental area becomes an isosceles triangle with base $r \, d\theta$ and height $r$. 

Let $y$ be the distance of centroid $g$ from x axis.

Here \[ y = \frac{2r}{3} \sin \theta \]

\[ W \] \[ K \] \[ T \]

\[ \bar{Y} = \frac{\int y \, dA}{A} \]

\[ A = \frac{\pi r^2}{2} \]

\[ \bar{Y} = \frac{\int y \, dA}{A} \]

\[ \bar{Y} = \frac{\int \frac{2r}{3} \sin \theta \, dA}{A} \]

\[ \bar{Y} = \frac{4r}{3\pi} \]

\[ \bar{X} = 0 \]
Centroid of a quarter circle

Let us consider a quarter circle with radius \( r \). Let \( O \) be the centre and \( G \) be the centroid of the quarter circle. Let us consider the \( x \) and \( y \) axes as shown in the figure.

Let us consider an elemental area \( dA \) with centroid \( g \) as shown in the figure.

Let \( y \) be the distance of centroid \( g \) from \( x \) axis. Neglecting the curvature, the elemental area becomes an isosceles triangle with base \( r \cdot d\theta \) and height \( r \).

Here \( y = \frac{2r}{3} \cdot \sin \theta \)

\[
\begin{align*}
\text{WKT} & \quad \bar{Y} = \frac{3}{\pi} \int y \cdot dA \\
Y & = \frac{3}{\pi} \int y \cdot dA \\
A & = \frac{\pi r^2}{2}
\end{align*}
\]

\[
\begin{align*}
\bar{Y} & = \frac{\int y \cdot dA}{A} \\
& = \frac{2r}{3} \cdot \sin \theta \cdot \frac{r^2}{2} \cdot d\theta
\end{align*}
\]

\[
\begin{align*}
3 & = \frac{2r}{\pi} \int \cos \theta \cdot \pi/2 \\
& = \frac{4r}{3\pi} \int [0,1] \\
\bar{Y} & = \frac{4r}{3\pi}
\end{align*}
\]

Similarly

\[
\bar{X} = \frac{4r}{3\pi}
\]
Centroid of Sector of a Circle

Consider the sector of a circle of angle $2\alpha$ as shown in Fig. Due to symmetry, centroid lies on $x$ axis. To find its distance from the centre $O$, consider the elemental area shown.

Area of the element $= r d\theta \, dr$

Its moment about $y$ axis

$= r d\theta \times dr \times r \cos \theta$

$= r^2 \cos \theta \, dr d\theta$

$\therefore$ Total moment of area about $y$ axis

$= \int_{-\alpha}^{\alpha} \int_{0}^{R} r^2 \cos \theta \, dr d\theta$

$= \left[ \frac{r^3}{3} \right]_{0}^{R} \left[ \sin \theta \right]_{-\alpha}^{\alpha}$

$= \frac{R^3}{3} \cdot 2 \sin \alpha$

Total area of the sector

$= \int_{-\alpha}^{\alpha} \int_{0}^{R} r \, dr d\theta$

$= \left[ \frac{r^2}{2} \right]_{0}^{R} \left[ \theta \right]_{-\alpha}^{\alpha}$

$= \frac{R^2}{2} \left[ \theta \right]_{-\alpha}^{\alpha}$

$= R^2 \alpha$

$\therefore$ The distance of centroid from centre $O$

$= \frac{\text{Moment of area about } y \text{ axis}}{\text{Area of the figure}}$

$= \frac{\frac{2R^3}{3} \sin \alpha}{\frac{R^2 \alpha}{3}} = \frac{2R}{3} \sin \alpha$
4.5 Centroid of Composite Sections

In engineering practice, use of sections which are built up of many simple sections is very common. Such sections may be called as built-up sections or composite sections. To locate the centroid of composite sections, one need not go for the first principle (method of integration). The given composite section can be split into suitable simple figures and then the centroid of each simple figure can be found by inspection or using the standard formulae listed in the table above. Assuming the area of the simple figure as concentrated at its centroid, its moment about an axis can be found by multiplying the area with distance of its centroid from the reference axis. After determining moment of each area about reference axis,
the distance of centroid from the axis is obtained by dividing total moment of area by total area of the composite section.

**PROBLEMS:**

Q) Locate the centroid of the T-section shown in fig.

![Diagram of T-section](image)

*Solution.* Selecting the axis as shown in Fig, we can say due to symmetry centroid lies on y axis, i.e. \( \bar{x} = 0 \). Now the given T-section may be divided into two rectangles \( A_1 \) and \( A_2 \) each of size \( 100 \times 20 \) and \( 20 \times 100 \). The centroid of \( A_1 \) and \( A_2 \) are \( g_1(0, 10) \) and \( g_2(0, 70) \) respectively.

\[
\bar{y} = \frac{100 \times 20 \times 10 + 20 \times 100 \times 70}{100 \times 20 + 20 \times 100} = 40 \text{ mm}
\]

Hence, centroid of T-section is on the symmetric axis at a distance 40 mm from the top.  

**Ans.**
Q) Find the centroid of the unequal angle 200×150×12 mm, shown in Fig.

Solution. The given composite figure can be divided into two rectangles:

\[ A_1 = 150 \times 12 = 1800 \text{ mm}^2 \]
\[ A_2 = (200 - 12) \times 12 = 2256 \text{ mm}^2 \]

Total area \( A = A_1 + A_2 = 4056 \text{ mm}^2 \)

Selecting the reference axis \( x \) and \( y \) as shown in Fig. 2.30. The centroid of \( A_1 \) is \( g_1 (75, 6) \)
and that of \( A_2 \) is:

\[ g_2 \left[ 6, \frac{1}{2} (200 - 12) \right] \]

i.e., \( g_2 (6, 106) \)

\[ \therefore \quad \bar{x} = \frac{\text{Movement about } y \text{ axis}}{\text{Total area}} \]
\[ = \frac{A_1 x_1 + A_2 x_2}{A} \]
\[ = \frac{1800 \times 75 + 2256 \times 6}{4056} = 36.62 \text{ mm} \]

\[ \bar{y} = \frac{\text{Movement about } x \text{ axis}}{\text{Total area}} \]
\[ = \frac{A_1 y_1 + A_2 y_2}{A} \]
\[ = \frac{1800 \times 6 + 2256 \times 106}{4056} = 61.62 \text{ mm} \]

Thus, the centroid is at \( \bar{x} = 36.62 \text{ mm} \) and \( \bar{y} = 61.62 \text{ mm} \) as shown in the figure.

\[ \text{Ans.} \]
Q) Locate the centroid of the I-section shown in Fig.

![Diagram of an I-section with dimensions labeled]

All dimensions in mm

Solution. Selecting the co-ordinate system as shown in Fig., due to symmetry centroid must lie on y axis,

*i.e.,* \( \bar{x} = 0 \)

Now, the composite section may be split into three rectangles

\[
A_1 = 100 \times 20 = 2000 \text{ mm}^2
\]

Centroid of \( A_1 \) from the origin is:

\[
y_1 = 30 + 100 + \frac{20}{2} = 140 \text{ mm}
\]

Similarly

\[
A_2 = 100 \times 20 = 2000 \text{ mm}^2
\]

\[
y_2 = 30 + \frac{100}{2} = 80 \text{ mm}
\]

\[
A_3 = 150 \times 30 = 4500 \text{ mm}^2, \text{ and}
\]

\[
y_3 = \frac{30}{2} = 15 \text{ mm}
\]

\[
\bar{y} = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A}
\]

\[
= \frac{2000 + 140 + 2000 \times 80 + 4500 \times 15}{2000 + 2000 + 4500}
\]

\[
= 59.71 \text{ mm}
\]

Thus, the centroid is on the symmetric axis at a distance 59.71 mm from the bottom as shown in Fig. 

Ans.
1. Determine the centroid of the lamina shown in fig. wrt O.

![Diagram of a lamina with dimensions and origin O]

<table>
<thead>
<tr>
<th>Component</th>
<th>Area (mm²)</th>
<th>X (mm)</th>
<th>Y (mm)</th>
<th>aX</th>
<th>aY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarter circle</td>
<td>-1256.64</td>
<td>16.97</td>
<td>16.97</td>
<td>-21325.2</td>
<td>-21325.2</td>
</tr>
<tr>
<td>Triangle</td>
<td>900</td>
<td>40</td>
<td>50</td>
<td>36000</td>
<td>45000</td>
</tr>
<tr>
<td>Rectangle</td>
<td>2400</td>
<td>30</td>
<td>20</td>
<td>72000</td>
<td>48000</td>
</tr>
<tr>
<td></td>
<td>∑a= 2043.36</td>
<td></td>
<td></td>
<td>∑aX = 86674.82</td>
<td>∑aY = 71674.82</td>
</tr>
</tbody>
</table>

X = 42.42 mm; Y = 35.08 mm

Find the centroid of the shaded area shown in fig, obtained by cutting a semicircle of diameter 100mm from the quadrant of a circle of radius 100mm. (Jan 2011)

![Diagram of a semicircle]

<table>
<thead>
<tr>
<th>Component</th>
<th>Area (mm²)</th>
<th>X (mm)</th>
<th>Y (mm)</th>
<th>aX</th>
<th>aY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarter circle</td>
<td>7853.98</td>
<td>42.44</td>
<td>42.44</td>
<td>333322.9</td>
<td>333322.9</td>
</tr>
<tr>
<td>Semi circle</td>
<td>-3926.99</td>
<td>50</td>
<td>21.22</td>
<td>-196350</td>
<td>-83330.7</td>
</tr>
<tr>
<td></td>
<td>∑a= 3926.99</td>
<td></td>
<td></td>
<td>∑aX = 136973.4</td>
<td>∑aY = 249992.2</td>
</tr>
</tbody>
</table>

X = 34.88 mm; Y = 63.66 mm
Where
\[ y = \text{distance from the x axis to area } dA \]
\[ x = \text{distance from the y axis to area } dA \]

**RADIUS OF GYRATION \( k \)**

The **radius of gyration** of an area with respect to a particular axis is the square root of the quotient of the moment of inertia divided by the area. It is the distance at which the entire area must be assumed to be concentrated in order that the product of the area and the square of this distance will equal the moment of inertia of the actual area about the given axis. In other words, the radius of gyration describes the way in which the total cross-sectional area is distributed around its centroidal axis. If more area is distributed further from the axis, it will have greater resistance to buckling. The most efficient column section to resist buckling is a circular pipe, because it has its area distributed as far away as possible from the centroid. Rearranging we have:

\[ I_x = k_x^2 A \]
\[ I_y = k_y^2 A \]

The radius of gyration is the distance \( k \) away from the axis that all the area can be concentrated to result in the same moment of inertia.

\[ k_x = \sqrt{\frac{I_x}{A}} \]
\[ k_y = \sqrt{\frac{I_y}{A}} \]

**Parallel Axis Theorem**

The moment of inertia of an area with respect to any given axis is equal to the moment of inertia with respect to the centroidal axis plus the product of the area and the square of the distance between the 2 axes.

The parallel axis theorem is used to determine the moment of inertia of composite sections.
Perpendicular Axis Theorem
Theorem of the perpendicular axis states that if $I_{XX}$ and $I_{YY}$ be the moment of inertia of a plane section about two mutually perpendicular axis X-X and Y-Y in the plane of the section, then the moment of inertia of the section $I_{ZZ}$ about the axis Z-Z, perpendicular to the plane and passing through the intersection of X-X and Y-Y is given by:

$$I_{ZZ} = I_{XX} + I_{YY}$$

The moment of inertia $I_{ZZ}$ is also known as polar moment of inertia.

Determination of the moment of inertia of an area by integration

The rectangular moments of inertia $I_x$ and $I_y$ of an area are defined as

$$I_x = \int x^2 \, dA \quad I_y = \int y^2 \, dA$$
1. **Rectangle** (Origin of axes at centroid.)

   \[ A = bh \]
   \[ \bar{x} = \frac{b}{2} \quad \bar{y} = \frac{h}{2} \]

   \[ I_x = \frac{bh^3}{12} \quad I_y = \frac{bh^3}{12} \]

   \[ I_{xy} = 0 \quad I_p = \frac{bh}{12} \left( b^2 + h^2 \right) \]

2. **Rectangle** (Origin of axes at corner.)

   \[ I_x = \frac{bh^3}{3} \quad I_y = \frac{bh^3}{3} \]

   \[ I_{xy} = \frac{b^3h^3}{4} \quad I_p = \frac{bh}{3} \left( h^2 + b^2 \right) \quad I_{pp} = \frac{b^3h^3}{6(b^2 + h^2)} \]

3. **Triangle** (Origin of axes at centroid.)

   \[ A = \frac{bh}{2} \]

   \[ x = \frac{b + c}{3} \quad y = \frac{h}{3} \]

   \[ I_x = \frac{bh^3}{36} \quad I_y = \frac{bh}{36} \left( b^2 - bc + c^2 \right) \]

   \[ I_{xy} = \frac{bh^2}{72} \left( b - 2c \right) \quad I_p = \frac{bh}{36} \left( b^2 + b^2 - bc + c^2 \right) \]

4. **Triangle** (Origin of axes at vertex.)

   \[ I_x = \frac{bh^3}{12} \quad I_y = \frac{bh}{12} \left( 3b^2 - 3bc + c^2 \right) \]

   \[ I_{xy} = \frac{bh^2}{24} \left( 3b - 2c \right) \quad I_{pp} = \frac{bh^3}{4} \]

5. **Isosceles triangle** (Origin of axes at centroid.)

   \[ A = \frac{bh}{2} \quad x = \frac{b}{2} \quad y = \frac{h}{3} \]

   \[ I_x = \frac{bh^3}{36} \quad I_y = \frac{bh^3}{48} \quad I_{xy} = 0 \]

   \[ I_p = \frac{bh}{144} \left( 4b^2 + 3h^2 \right) \quad I_{pp} = \frac{bh^3}{12} \]

(Note: For an equilateral triangle, \( h = \sqrt{3}b/2 \).)
Right triangle (Origin of axes at centroid.)

\[ A = \frac{bh}{2} \quad \bar{x} = \frac{b}{3} \quad \bar{y} = \frac{h}{3} \]

\[ I_x = \frac{bh^3}{36} \quad I_y = \frac{h^3b}{36} \quad I_{xy} = -\frac{b^2h^2}{72} \]

\[ I_p = \frac{bh}{36} (h^2 + b^2) \quad I_{BB} = \frac{bh^3}{12} \]

Right triangle (Origin of axes at vertex.)

\[ I_x = \frac{bh^3}{12} \quad I_y = \frac{h^3b}{12} \quad I_{xy} = \frac{b^2h^2}{24} \]

\[ I_p = \frac{bh}{12} (h^2 + b^2) \quad I_{BB} = \frac{bh^3}{4} \]

Trapezoid (Origin of axes at centroid.)

\[ A = \frac{h(a + b)}{2} \quad \bar{y} = \frac{h(2a + b)}{3(a + b)} \]

\[ I_x = \frac{h^2(a^2 + 4ab + b^2)}{36(a + b)} \quad I_{BB} = \frac{h^3(3a + b)}{12} \]

Circle (Origin of axes at center.)

\[ A = \pi r^2 = \frac{\pi d^2}{4} \quad I_x = I_y = \frac{\pi r^4}{4} = \frac{\pi d^4}{64} \]

\[ I_{xy} = 0 \quad I_p = \frac{\pi r^4}{2} = \frac{\pi d^4}{32} \quad I_{BB} = \frac{5\pi r^4}{4} = \frac{5\pi d^4}{64} \]

Circular ring (Origin of axes at center.)

Approximate formulas for case when \( t \) is small.

\[ A = 2\pi rt = \pi d t \quad I_x = I_y = \pi r^3 t = \frac{\pi d^3 t}{8} \]

\[ I_{xy} = 0 \quad I_p = 2\pi r^3 t = \frac{\pi d^3 t}{4} \]
Semicircle (Origin of axes at centroid.)

\[ A = \frac{\pi r^2}{2}, \quad \bar{y} = \frac{4r}{3\pi} \]
\[ I_x = \frac{(9\pi^2 - 64)r^4}{72\pi} \approx 0.1098r^4, \quad I_y = \frac{\pi r^4}{8} \]
\[ I_{xy} = 0, \quad I_{BB} = \frac{\pi r^4}{8} \]

Quarter circle (Origin of axes at center of circle.)

\[ A = \frac{\pi r^2}{4}, \quad \bar{x} = \bar{y} = \frac{4r}{3\pi} \]
\[ I_x - I_y = \frac{\pi r^4}{16}, \quad I_{xy} = \frac{r^4}{8} \]
\[ I_{BB} = \frac{(9\pi^2 - 64)r^4}{144\pi} \approx 0.05488r^4 \]

Quarter-circular spandrel (Origin of axes at vertex.)

\[ A = \left(1 - \frac{\pi}{4}\right)r^2 \]
\[ \bar{x} = \frac{2r}{3(4 - \pi)} \approx 0.7766r, \quad \bar{y} = \frac{(10 - 3\pi)r}{3(4 - \pi)} \approx 0.2234r \]
\[ I_x = \left(1 - \frac{5\pi}{16}\right)r^4 \approx 0.01825r^4, \quad I_y = I_{BB} = \left(\frac{1}{3} - \frac{\pi}{16}\right)r^4 \approx 0.1370r^4 \]

Example

Determine the moment of inertia of the shaded area shown with respect to each of the coordinate axes.
**Moment of Inertia \( I_x \)**

\[
I_x = \int_A y^2 \, dA
= \int_0^b y^2 (a-x) \, dy
= a \left[ \frac{y^3}{3} \right]_0^b - \frac{a}{b^{1/2}} \left[ \frac{y^{3/2}}{2} \right]_0^b
= ab^3 - \frac{a}{b^{1/2}} \left( \frac{2}{7} b^{3/2} \right)
= \frac{ab^3}{3} - \frac{2ab^3}{7}
= \frac{ab^3}{21}
\]

Substituting \( x = a \) and \( y = b \)

\[
y = ka^2
b = ka^2
k = \frac{b}{a^2}
\]

\[
y = \frac{b}{a^2} x^2 \text{ or } x = \frac{a}{b^{1/2}} y^{1/2}
\]

**Moment of Inertia \( I_y \)**

\[
I_y = \int_A x^2 \, dA
= \int_0^a x^2 \, y \, dx
= \int_0^a x^2 \left( \frac{b}{a^2} x^2 \right) \, dx
= \frac{b}{a^2} \int_0^a x^4 \, dx
= \left( \frac{b}{a^2} \right) \left( \frac{x^5}{5} \right) \bigg|_0^a
= \left( \frac{b}{a^2} \right) \left( \frac{a^5}{5} \right)
= \frac{a^3 b}{5}
\]

\[\text{or}\]

\[
= \frac{a^3 b}{5}
\]
Moment of Inertia $I_y$

\[ I_y = \int x^2 dA \]
\[ = \int_0^a x^2 (y_1 - y_2) dx \]
\[ = \int_0^a x^2 (x - x^2) dx \]
\[ = \int_0^a (x^3) dx - \int_0^a (x^4) dx \]
\[ = \frac{x^4}{4} \bigg|_0^a - \frac{x^5}{5} \bigg|_0^a \]
\[ = \frac{a^4}{4} - \frac{a^5}{5} \]

Moment of inertia of composite areas

A similar theorem can be used with the polar moment of inertia. The polar moment of inertia $J_0$ of an area about $O$ and the polar moment of inertia $J_c$ of the area about its centroid are related to the distance $d$ between points $C$ and $O$ by the relationship

\[ J_0 = J_c + Ad^2 \]

The parallel-axis theorem is used very effectively to compute the moment of inertia of a composite area with respect to a given axis.
Example

Compute the moment of inertia of the composite area shown.

\[ I_x = \left( \frac{bh^3}{3} \right)_{\text{Rect}} - \left( I_x + Ad_y^2 \right)_{\text{Cir}} \]

\[ = \left[ \frac{1}{3} (100)(150)^3 \right]_{\text{Rect}} - \left[ \frac{1}{4} \pi (25)^4 + (\pi \times 25^2)(75)^2 \right]_{\text{Cir}} \]

\[ = 101 \times 10^6 \text{ mm}^4 \]
Example

Determine the moments of inertia of the beam’s cross-sectional area shown about the $x$ and $y$ centroidal axes.

Dimension in mm
Example
Determine the moments of inertia and the radius of gyration of the shaded area with respect to the $x$ and $y$ axes.

**SOLUTION**

\[
I_x = (\bar{I}_x + Ad_y^2)_A + (\bar{I}_x + Ad_y^2)_B + (\bar{I}_x + Ad_y^2)_C
\]

\[
= \left[ \frac{1}{12} (24)(6)^3 + (24 \times 6)(27)^2 \right]_A + \left[ \frac{1}{12} (8)(48)^3 + 0 \right]_B + \left[ \frac{1}{12} (48)(6)^3 + (48 \times 6)(27)^2 \right]_C
\]

\[
I_x = 390 \times 10^3 \text{ mm}^4
\]

\[
k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{390 \times 10^3}{[(24 \times 6) + (8 \times 48) + (48 \times 6)]}} = 21.9 \text{ mm}
\]

\[
I_y = (\bar{I}_y + Ad_x^2)_A + (\bar{I}_y + Ad_x^2)_B + (\bar{I}_y + Ad_x^2)_C
\]

\[
= \left[ \frac{1}{12} (6)(24)^3 \right]_A + \left[ \frac{1}{12} (48)(8)^3 \right]_B + \left[ \frac{1}{12} (6)(48)^3 \right]_C
\]

\[
I_y = 64.3 \times 10^3 \text{ mm}^4
\]

\[
k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{64.3 \times 10^3}{[(24 \times 6) + (8 \times 48) + (48 \times 6)]}} = 8.87 \text{ mm}
\]
Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes.

\[ I_x = \left( I_x + A d_{y}^2 \right)_{A} - \left( I_x + A d_{y}^2 \right)_{B} + \left( I_x + A d_{y}^2 \right)_{C} \]

\[ = \frac{1}{12} \cdot \frac{(5)(6)^2}{2} + 0 \] _A_ - \left[ \frac{1}{12} \cdot (4)(2)^2 + (2 \times 4)(2)^2 \right] _B

\[ - \left[ \frac{1}{12} \cdot (4)(1)^2 + (4 \times 1)(2)^2 \right] _C \]

\[ I_x = 46 \text{ m}^4 \]

\[ k_x = \frac{I_x}{A} = \sqrt{\frac{46}{(5 \times 6) - (4 \times 2) - (4 \times 1)}} = 1.599 \text{ m} \]

\[ I_y = \left( I_y + A d_{x}^2 \right)_{A} - \left( I_y + A d_{x}^2 \right)_{B} + \left( I_y + A d_{x}^2 \right)_{C} \]

\[ = \left[ \frac{1}{12} \cdot (6)(5)^2 \right] _A - \left[ \frac{1}{12} \cdot (2)(4)^2 \right] _B - \left[ \frac{1}{12} \cdot (1)(4)^2 \right] _C \]

\[ I_y = 46.5 \text{ m}^4 \]

\[ k_y = \frac{I_y}{A} = \sqrt{\frac{46.5}{(5 \times 6) - (4 \times 2) - (4 \times 1)}} = 1.607 \text{ m} \]

**Example**
Example

Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes and at the centroidal axes.

![Diagram of U-shaped object with dimensions and calculations]

- Moments of inertia about centroid
  \( \bar{I}_x = I_x - Ad_y^2 \)
  \[ = 145 - (15)(2.5)^2 \]
  \[ = 51.25 \text{ cm}^4 \]

  OR
  \[ \bar{I}_x = 2\left[\frac{1}{12}(1)(5)^3 + (5 \times 1)(1)^3\right] \]
  \[ + \left[\frac{1}{12}(5)(1)^3 + (5 \times 1)(2)^3\right] \]
  \[ = 51.25 \text{ cm}^4 \]

- Moments of inertia about x axis
  \( I_x = 2\left[\frac{1}{12}(1)(5)^3 + (5 \times 1)(3.5)^2\right] + \frac{1}{3}(5)(1)^3 \)
  \[ = 145 \text{ cm}^4 \]

- \( \bar{k}_x = \bar{k}_y = \sqrt{\frac{\bar{I}_x}{A}} = \sqrt{\frac{51.25}{15}} = 1.848 \text{ cm} \)
Example

The strength of a W360 x 57 rolled-steel beam is increased by attaching a 229 mm x 19 mm plate to its upper flange as shown. Determine the moment of inertia and the radius of gyration of the composite section with respect to an axis which is parallel to the plate and passes through the centroid C of the section.

**SOLUTION**

**Centroid**

The wide-flange shape of W360 x 57 found by referring to Fig. 9.13

\[ A = 7230 \text{ mm}^2 \quad \bar{I}_x = 160.2 \text{ mm}^4 \]

\[ A_{\text{plate}} = (229)(19) = 4351 \text{ mm}^2 \]

\[ \overline{yA} = \sum yA \]

\[ \overline{y}(4351 + 7230) = (188.5)(4351) + (0)(7230) \]

\[ \overline{y} = 70.8 \text{ mm} \]

**Moment of Inertia**

\[ I_x = (I_x)_{\text{plate}} + (I_x)_{\text{wide-flange}} \]

\[ = (\bar{I}_x + A\bar{d}^2)_{\text{plate}} + (I_x + A\bar{y}^2)_{\text{wide-flange}} \]

\[ = \left[ \frac{1}{12} (229)(19)^3 + (4351)(188.5 - 70.8)^2 \right] \]

\[ + \left[ 60.2 \times 10^6 + (7230)(70.8)^2 \right] \]

\[ = 256.8 \times 10^6 \text{ mm}^4 \]

\[ I_x = 257 \times 10^6 \text{ mm}^4 \]

**Radius of Gyration**

\[ k_x = \frac{I_x}{A} = \frac{256.8 \times 10^6}{4351 + 7230} \]

\[ k_y = 149 \text{ mm} \]
Polar Moment of Inertia

The polar moment of inertia of an area \( A \) with respect to the pole \( O \) is defined as

\[
J_o = \int r^2 \, dA
\]

The distance from \( O \) to the element of area \( dA \) is \( r \). Observing that \( r^2 = x^2 + y^2 \), we established the relation

\[
J_o = I_x + I_y
\]

Example

(a) Determine the centroidal polar moment of inertia of a circular area by direct integration. (b) Using the result of part a, determine the moment of inertia of a circular area with respect to a diameter.